

$$= a^2 v_x(x,t) v_t(x,t) \Big|_0^l + \int_0^l a^2 v_{xx}(x,t) v_t(x,t) + v_t(x,t) v_{tt}(x,t) dx =$$

$$= a^2 v_x(l,t) v_t(l,t) - a^2 v_x(0,t) v_t(0,t) + \int_0^l v_t(x,t) [-a^2 v_{xx} + v_{tt}(x,t)] dx =$$

$$= 0$$

$$\frac{dE}{dt}(t) = 0$$

$$E(t) = \text{const}, t \in [0, T]$$

$$E(0) = \frac{1}{2} \int_0^l [a^2 (v_x(x,0))^2 + (v_t(x,0))^2] dx = 0 \quad (\text{из I и II краевых условий}), \text{возвращаясь к первоначальному заданию, имеем}$$

$$v_x(x,t) = v_t(x,t) = 0 \quad \forall t$$

$$v(x,t) = \text{const}, \text{ а поскольку имеем начальные краевые условия, получаем,}$$

$$\text{что это } 0.$$

Для любой задачи доказательства suffice.

Задача №15 (Ильинский)

$$\begin{cases} u_t + uu_x = 0 \frac{\partial^2 u}{\partial x^2}, & -\infty < x < +\infty, t > 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$

$$u = -20 \ln \varphi(x), \quad u_t = -20 (\ln \varphi)_x$$

$$\begin{cases} \frac{\partial v}{\partial t} = 0 \frac{\partial^2 v}{\partial x^2}, & -\infty < x < +\infty \\ v|_{t=0} = \varphi(x) \end{cases}$$

$$v|_{t=0} = \varphi(x)$$

$$\text{Пусть } \varphi(x) = e^{-x^2}$$

$$e^{-x^2} = -20 (\ln \varphi)_x$$

$$-\frac{1}{20} \int_0^x e^{-\eta^2} d\eta = \ln \varphi(x)$$

$$v|_{t=0} = e^{-\frac{\sqrt{\pi}}{40} \cdot \text{erf}(x)}$$

$$\begin{aligned} \varphi(x) &= -20 (\ln \varphi)_x \\ (\ln \varphi)_x &= -\frac{\varphi(x)}{20} \\ \ln \varphi &= -\frac{1}{20} \int_0^x \varphi(\eta) d\eta \\ \varphi(x) &= e^{-\frac{1}{20} \int_0^x \varphi(\eta) d\eta} \end{aligned}$$

$$\begin{aligned} \varphi(x) &= -20 (\ln \varphi)_x \\ (\ln \varphi)_x &= -\frac{\varphi(x)}{20} \end{aligned}$$

$$\ln \varphi = -\frac{1}{20} \int_0^x \varphi(\eta) d\eta$$

$$\varphi(x) = e^{-\frac{1}{20} \int_0^x \varphi(\eta) d\eta}$$

$$u(x,t) = \frac{1}{2\sqrt{D\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} e^{-\frac{\sqrt{t}}{4D}} \phi(y) dy$$

Уравнение Корнелиуса-го Фриша КГБ

$$u_t + 6uu_x + u_{xxx} = 0$$

$$x' = kx, t' = \eta t, u' = \alpha u$$

$$u = f(x - ct) = f(y)$$

$$u_t = -cf'_y$$

$$u_x = f'_y$$

$$u_{xxx} = f'''_{yyy}$$

$$-cf'_y + f'_y \cdot 6 + f'''_{yyy} = 0$$

$$\frac{-cf}{2} + \frac{6}{2} f^2 + f''_{yy} = A$$

$$-cf f' + 3f^2 f' + f'$$

$$-cf f' + 3f^2 f' + f' f' = A f' \quad (\text{умножив на } y)$$

$$-\frac{c}{2} f^2 + f^3 + \frac{1}{2} (f')^2 = A f + B$$

$$\text{Положим } A=0, B=0$$

$$-cf^2 + 2f^3 + (f')^2 = 0$$

$$(f')^2 = cf^2 - 2f^3$$

$$f' = \pm f \sqrt{c - 2f}$$

$$\frac{df}{dy} = \pm f \sqrt{c - 2f}$$

$$dy = \pm \frac{df}{f \sqrt{c - 2f}}$$

$$\sqrt{c} dy = \pm \frac{df}{f \sqrt{1 - \frac{2f}{c}}}$$

$$\sqrt{c}(y - y_0) = \pm \int \frac{df}{f \sqrt{1 - \frac{2f}{c}}}$$

$$\frac{2}{c} f = \frac{1}{\cosh^2 x}$$

$$1 - \frac{1}{\cosh^2 x} = \frac{\cosh^2 x - 1}{\cosh^2 x} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x$$

$$f = \frac{c}{2} \cdot \frac{1}{\cosh^2 x}$$

$$df = \frac{c}{2} \cdot \frac{-2 \cdot \sinh x}{\cosh^3 x} dx = -c \tanh x \cdot \frac{1}{\cosh^2 x} dx$$

$$\sqrt{c}(y-x_0) = \pm \int \frac{c \tanh x dx}{\cosh^2 x \cdot \frac{c}{2 \cosh^2 x} \sinh x}$$

$$\frac{\sqrt{c}}{2}(y-x_0) = \pm \int dx$$

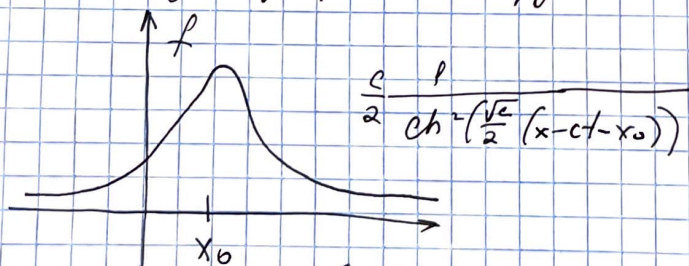
$$\cosh\left(\frac{\sqrt{c}}{2}(y-x_0)\right) = \cosh x$$

$$\cosh^2\left(\frac{\sqrt{c}}{2}(y-x_0)\right) = \cosh^2 x = \frac{c}{2f}$$

$$\frac{2}{c} f = \frac{1}{\cosh^2\left(\frac{\sqrt{c}}{2}(y-x_0)\right)}$$

$$u(x,t) = f(x-ct) = \frac{c}{2} \frac{1}{\cosh^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right)}$$

Нарисуем график этой функции.



Столкнулись две волны,

Уравнение Кошикова - Менина

Нелинейное ур-ние теплопроводности.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right), & 0 < x < 1, t > 0 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

Уравнение Д'Алессандро

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + f(u)$$

$$k(u) = u^3$$

$$f(u) = u^3$$

$$u = x(x) \tau(t)$$

$$x \tau' = (x x')' \tau^2$$

$$\frac{\tau'}{\tau^2} = \frac{(x x')'}{x} = -\lambda$$

$$\begin{cases} (x x')' + \lambda x \\ x(0) = 0; x(1) = 0 \end{cases}$$

$$x = \alpha \tilde{x}$$

$$\alpha^2 (\tilde{x} \tilde{x}')' + \lambda \alpha \tilde{x} = 0$$

$$\alpha = \lambda$$

$$\begin{cases} (\tilde{x} \tilde{x}')' + \tilde{x} = 0 \\ \tilde{x}(0) = 0; \tilde{x}(1) = 0 \end{cases}$$

$$\tilde{x} = y$$

$$\begin{cases} (y y')' + y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

$$\left(\frac{\sqrt{g} g'}{2\sqrt{g}} \right)' + \sqrt{g} = 0$$

$$y = F(x)$$

$$y^2 = g$$

$$g'' + 2\sqrt{g} = 0$$

$$g'' = -2\sqrt{g}$$

$$\frac{\tau'}{\tau^2} = -1; \frac{d\tau}{\tau^2} = -dt; \tau(t) = \frac{1}{t+t_0}$$

$$u(x,t) = \frac{F(x)}{t+t_0}$$

$$u(x,t) = \varphi_1 \sin \pi x e$$

$$-(\pi a)^2 t$$

$$+ \varphi_2 \sin 2\pi x e$$

$$-(\pi a)^2 t$$

$$u(x,t) \sim \varphi_1 \sin \pi x e$$

$$-\varphi(\pi a)^2 t$$

Лекция №12 (Ильинский)

Энергия колебаний струны

$$P = \frac{1}{2} \int_0^l \rho u_t^2 dx$$

$$u_{tt} = u_{xx}$$

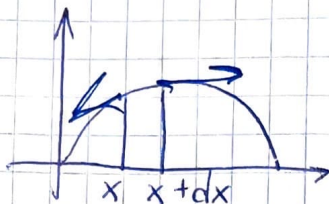
$$\cos \alpha = \frac{1}{\sqrt{1+u_x^2}}, \quad u_x^2 \ll 1$$

$$[T]_x = T \cos \alpha \approx T$$

$$[T]_{x+dx} = T \sin \alpha \approx T \sin \alpha = T u_x$$

$$P(x_1) = P(x_2)$$

$$\sin \alpha = \frac{u_x}{\sqrt{1+u_x^2}}$$



$$\rho u_{tt} dx = T u_x \big|_{x+dx} - T u_x \big|_x + F(x,t) dx$$

$$\rho u_{tt} dx = \frac{\partial}{\partial x} (T u_x) dx + F(x,t) dx$$

$$\rho u_{tt} = \frac{\partial}{\partial x} T \frac{\partial u}{\partial x} + F(x,t)$$

$$\rho = \rho_0 = \text{const}, \quad T(x) = T_0 = \text{const}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)}$$

$$\rho u_{tt} dx = T u_x \big|_{x+dx} - T u_x \big|_x + F(x,t) dx$$

$$+ F(x,t) dx$$

$$T u_x \big|_{x+dx} - T u_x \big|_x =$$

$$f(x,t) = \frac{F(x,t)}{\rho_0} = T u_{xx} dx$$

Энергия колебаний струны

$$P = \frac{1}{2} \int_0^l \rho u_t^2 dx - \text{кинетическая энергия}$$

Получим производную, которую застроим в уравнение

$$\left(\int_0^l T u_{xx} u_t dx \right) dt = \left(\int_0^l T u_t dx \right) dt = \left(T u_t u_x \big|_0^l - \int_0^l T u_x u_{xt} dx \right) dt$$

$$= \left(- \frac{d}{dt} \int_0^l \frac{T u_x^2}{2} dx \right) dt$$

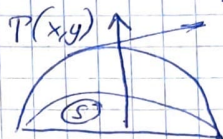
$$A = - \int_0^L \int_0^t \left(\frac{1}{2} \frac{d}{dt} P u_x^2 \right) dx = - \frac{1}{2} \int_0^L P u_x^2 dx =$$

-U

$$U = \int_0^L \frac{1}{2} P u_x^2 dx$$

$$E(t) = \frac{1}{2} \int_0^L (\rho u_t^2 + P u_x^2) dx$$

Рассмотрим колебания плоской мембраны.



$$E(t) = \frac{1}{2} \int_0^L (\rho u_t^2 + P u_x^2) dx$$

$$\iint_S \rho u_{tt} dx dy = \oint \rho \frac{\partial u}{\partial n} dS + \iint_S F(x,y,t) dx dy;$$

$$\iint_S \rho u_{tt} dx dy = \iint_S \operatorname{div}(\rho \operatorname{grad} u) dx dy + \iint_S F(x,y,t) dx dy$$

$$\rho u_{tt} = \operatorname{div}(\rho \operatorname{grad} u) + F(x,y,t)$$

Если $P = P_0 = \text{const}$, $\rho = \rho_0 = \text{const}$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u + f(x,y,t), \quad f = \frac{F}{\rho_0}$$

$$a^2 = \frac{P}{\rho_0}$$

Тензорные уравнения

$$I(x,t), \quad x, \quad x+dx$$

R, L, C, G - коэф-ты в 1 м длины

$$v(x) - v(x+dx) = -v_x dx$$

$$-v_x dx = R i dx + L i \frac{dv}{dt} dx$$

$$v_x + R i + L i \frac{dv}{dt} = 0 \quad (1)$$

Закон сохранения заряда.

$$(i(x,t) - i(x+dx,t)) dt = C dx (v(x,t+dt) - v(x,t)) + G v dx dt$$

$$-i_x = C v_t + G v \quad (2)$$

$L dx$ - индуктивность отрезка dx

Уравнения в

$$\begin{cases} V_x + Ri + Li_t = 0 \\ i_x + Ci_t + GV = 0 \end{cases}$$

Уравнения в

$$\begin{cases} V_{x+} + Ri_t + Li_{t+} = 0 \\ i_{xx} + (V_{x+} + GV_x) = 0 \end{cases}$$

$$RCi_t + CLi_{t+} - i_{xx} - GV_x = 0$$

$$i_{xx} = LCi_{t+} + i_t(RC + GL) + G Ri = 0$$

Если $R=0, G=0$, то

$$i_{t+} = \frac{1}{LC} i_{xx}$$

$$V_{t+} = \frac{1}{LC} V_{xx}$$

Уравнения Максвелла

$$\begin{cases} \text{rot } \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \text{rot } \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \\ \text{div } \vec{E} = 0 \\ \text{div } \vec{B} = 0 \end{cases}$$

$$\vec{E} = \vec{E}(x, y, z, t)$$

$$\vec{B} = \vec{B}(x, y, z, t)$$

$$\text{rot}(\text{rot } \vec{E}) = -\frac{1}{c} \frac{\partial}{\partial t} (\text{rot } \vec{B})$$

$$\text{rot}(\text{rot } \vec{E}) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{rot}(\text{rot } \vec{A}) = \text{grad}(\text{div } \vec{A}) - \text{div}(\text{grad } \vec{A})$$

$$\text{grad}(\text{div } \vec{E}) - \text{div}(\text{grad } \vec{E}) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$-\Delta \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{Если } \vec{E} = \{E_x, E_y, E_z\}, \text{ то}$$

$$\Delta E_1 = \frac{1}{c^2} \frac{\partial^2 E_1}{\partial t^2}$$

$$\Delta E_2 = \frac{1}{c^2} \frac{\partial^2 E_2}{\partial t^2}$$

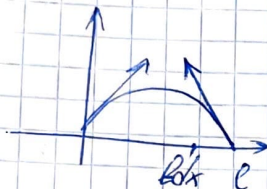
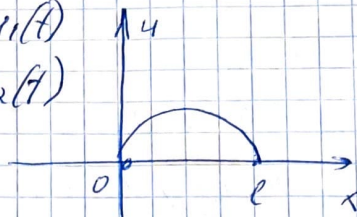
$$\Delta E_3 = \frac{1}{c^2} \frac{\partial^2 E_3}{\partial t^2}$$

Трехмерные условия.

$$u_{tt} = a^2 u_{xx} \quad u(0,t) = u_1(t)$$

$$u(0,t) = 0 \quad u(l,t) = u_2(t)$$

$$u(l,t) = 0$$



$$\rho dx \cdot u_{tt} = -T u_x|_{x=l} - h u(l,t)$$

$$\lim_{x \rightarrow 0} u_x = 0$$

$$0 = (-T u_x - h u)|_{x=l}$$

$$-T u_x + h u|_{x=0} = 0$$

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x); \\ u_t(x,0) = \psi(x) \end{cases}$$

$$u(x,t) \in C^2(t > 0) \cap C^1(t \geq 0)$$

Решаем:

$$dx^2 - a^2 dt^2 = 0$$

$$\begin{cases} x + at = c_1 \\ x - at = c_2 \end{cases}$$

$$\begin{cases} x + at = c_1 \\ x - at = c_2 \end{cases}$$

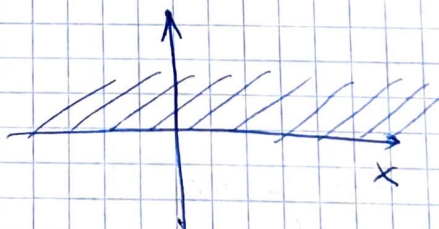
$$\eta = x + at,$$

$$\eta = x - at$$

$u(\eta) = 0$ после приведения к каноническому виду.

$$\frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \eta} \right) = 0$$

$$\frac{\partial u}{\partial \eta} = C_1 \eta(y)$$



$$u(y, \eta) = \tilde{c}_1(y) + c_2(\eta)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \quad \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = 0$$

$$\frac{\partial u}{\partial y} = G(y)$$

$$u(x, y) = \tilde{c}_1(y) + c_2(x)$$

$$u(x, y) = f_1(x+at) + f_2(x-at)$$

$$x+at = \text{const}$$

$$\frac{dx}{dt} = -a$$

$$x-at = \text{const}, \quad \frac{dx}{dt} = a, \quad \text{удовлетворяем начальным условиям.}$$

$$t=0 \quad \int f_1(x) + f_2(x) = \varphi(x)$$

$$\begin{cases} a f_1'(x) - a f_2'(x) = \psi(x), & f_1(x) - f_2(x) = \int_{x_0}^x \frac{1}{a} \psi(\alpha) d\alpha + C \end{cases}$$

$$\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ f_1(x) - f_2(x) = \frac{1}{a} \int_{x_0}^x \psi(\alpha) d\alpha + C \end{cases}$$

$$f_1(x) = \frac{\varphi(x)}{2} + \frac{1}{2a} \int_{x_0}^x \psi(\alpha) d\alpha + \frac{C}{2}$$

$$f_2(x) = \frac{\varphi(x)}{2} - \frac{1}{2a} \int_{x_0}^x \psi(\alpha) d\alpha - \frac{C}{2}$$

$$u(x, t) = \frac{\varphi(x+at)}{2} + \frac{\varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha$$

$$u(x, t) = \frac{\varphi(x+at)}{2} + \frac{\varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha$$

$$\text{Пусть } |\psi| \leq \varepsilon, \quad |\varphi| \leq \varepsilon$$

$$|u| \leq \frac{|\varphi(x+at)|}{2} + \frac{|\varphi(x-at)|}{2} + \frac{1}{2a} \int_{x-at}^{x+at} |\psi(\alpha)| d\alpha \leq$$

$$\leq \varepsilon (1 + \frac{1}{a})$$

$$\begin{cases} u_{tt} = -u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x, 0) = 0 \\ u_t(x, 0) = 0 \\ u(x, t) = 0 \end{cases}$$

$$\begin{cases} u_{tt} = -u_{xx}, \\ u(x, 0) = \varepsilon \sin kx \\ u_t(x, 0) = 0 \end{cases}$$

$$u(x, t) = \varepsilon \sin kx \cos(kt) \quad \begin{matrix} t \rightarrow 0 \\ t \rightarrow \infty \end{matrix}$$

$$\begin{cases} u_{xx} - u_{yy} = 0 & 0 < x, y < \pi \\ u|_r = 0 & u(x, y) = C \sin x \sin y \end{cases}$$

Полупроводниковая проволока.

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, t > 0 \\ u(0, t) = u(t) \\ u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 (u_1)_{xx} \\ u_1(0, t) = 0, \\ u_1(x, 0) = \varphi(x), x > 0 \\ u_{1t}(x, 0) = \psi(x), x > 0 \end{cases}$$

$$u_1(x, t) = f_1(x+at) + f_2(x-at)$$

$$u(x, t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha, \quad x-at > 0$$

or zero system, else $x-at < 0$

$$u_1(0, t) = f_1(at) + f_2(-at) = 0$$

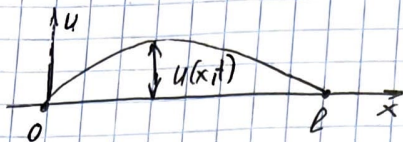
$$f_2(z) = -f_1(z)$$

$$u(x, t) = \frac{\varphi(x+at) - \varphi(at-x)}{2} + \frac{1}{2a} \int_{at-x}^{x+at} \psi(\alpha) d\alpha.$$

$$u(x, t)$$

$$u_{tt} = a^2 u_{xx} + f(x, t)$$

Колебание тока и ток гармон.



Задача на отрезке $x \in [0, l]$

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, \quad 0 < t \leq T \\ u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T \\ u(x, 0) = \varphi(x), \quad 0 \leq x \leq l \\ u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l \end{cases}$$

И краевые условия для уравнения поперечного

Краевые условия:

1) $u(0, t) = \mu(t), \quad 0 \leq t \leq T$

Второе краевое условие

2) $u_x(0, t) = \nu(t), \quad 0 \leq t \leq T$

Третье краевое условие

3) $\mu u(0, t) + \lambda u_x(l, t) = \theta(t), \quad 0 \leq t \leq T$

Начальные условия

$$u(x, 0) = \varphi(x) \quad 0 \leq x \leq l$$

$$u_t(x, 0) = \psi(x)$$

Первое краевое: задача для уравнения колебаний на отрезке.

Ищем $u(x, t)$

$$u_{tt} = a^2 u_{xx} + f(x, t), \quad 0 < x < l, \quad 0 < t \leq T,$$

$$\begin{cases} u(0, t) = \mu_1(t), \quad 0 \leq t \leq T \\ u(l, t) = \mu_2(t), \quad 0 \leq t \leq T \\ u(x, 0) = \varphi(x), \quad 0 \leq x \leq l \\ u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l \end{cases}$$

Второе краевое задание для уравнения колебаний на отрезке.

Найти $u(x, t)$, т.ч.

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & 0 < x < l, \\ u_x(0, t) = \gamma_1(t), & 0 \leq t \leq T, \\ u_x(l, t) = \gamma_2(t), & 0 \leq t \leq T, \\ u(x, 0) = \varphi(x), & 0 \leq x \leq l, \\ u_t(x, 0) = \psi(x), & 0 \leq x \leq l \end{cases}$$

Задача на полуинтервале

Первое краевое задание

Найти $u(x, t)$ такую, что

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & x > 0, \\ u(0, t) = \mu(t), & 0 \leq t \leq T, \\ u(x, 0) = \varphi(x), & x \geq 0, \\ u_t(x, 0) = \psi(x), & x \geq 0 \end{cases}$$

Второе краевое задание

Найти $u(x, t)$ такую, что

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & x > 0, \\ u_x(0, t) = \mu(t), & 0 \leq t \leq T, \\ u(x, 0) = \varphi(x), & 0 \leq x < l, \\ u_t(x, 0) = \psi(x), & 0 \leq x \leq l \end{cases}$$

Задача Коши для уравнения колебаний

Найти $u(x, t)$ такую, что

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & -\infty < x < \infty, \\ u(x, 0) = \varphi(x), & -\infty < x < \infty, \\ u_t(x, 0) = \psi(x), & -\infty < x < \infty \end{cases}$$

Задача Коши для уравнения колебаний. Формула Даламбера.

$$\begin{cases} u_{tt}(x, t) = a^2 u_{xx} & , -\infty < x < \infty & (1) \\ u(x, 0) = \varphi(x) & & (2) \\ u_t(x, 0) = \psi(x), & -\infty < x < \infty & (3) \end{cases}$$

$a^2, \varphi(x), \psi(x)$ заданы

Найти $u(x, t)$ удовн. (1)-(3)

Пусть $u(x,t)$ глобально непрерывно групп-и и упробн. (1)-(3)

$$\eta = x + at$$

$$\eta = \mu u$$

$$x = \frac{\eta + \eta}{2}$$

$$\eta = x - at$$

$$\eta = \mu a$$

$$t = \frac{\eta - \eta}{2a}$$

$$v(\eta, \eta) = u\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right)$$

$$v_\eta(\eta, \eta) = u_x\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \frac{1}{2} + u_t\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \left(\frac{1}{2a}\right)$$

$$v_{\eta\eta}(\eta, \eta) = u_{xx}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \frac{1}{4} + u_{xt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \left(\frac{1}{4a}\right) + u_{xt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \frac{1}{4a} + u_{tt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \left(\frac{1}{4a^2}\right) = \frac{1}{4} u_{xx}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) + \frac{1}{2a} u_{xt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) + u_{tt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) \cdot \frac{1}{4a^2} = \frac{1}{4a^2} [a^2 u_{xx}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right) - u_{tt}\left(\frac{\eta + \eta}{2}, \frac{\eta - \eta}{2a}\right)] = 0$$

$$v_{\eta\eta}(\eta, \eta) = 0$$

$$v_\eta(\eta, \eta) = f_1(\eta) +$$

$$v(\eta, \eta) = \int f_1(\eta) d\eta + f_2(\eta)$$

$$v(\eta, \eta) = f_1(\eta) + f_2(\eta)$$

$$u(x,t) = v(x+at, x-at) = f_1(x-at) + f_2(x+at)$$

$$u(x,0) = f_1(x) + f_2(x) = \varphi(x)$$

$$u_t(x,0) = -a f_1'(x) + a f_2'(x) = \psi(x)$$

$$-f_1(x) + f_2(x) = \frac{1}{a} \int_{x_0}^x \psi(s) ds + C \quad (*)$$

$$2f_2(x) = \varphi(x) + \frac{1}{a} \int_{x_0}^x \psi(s) ds + C$$

$$f_2(x) = \frac{\varphi(x)}{2} + \frac{1}{2a} \int_{x_0}^x \psi(s) ds + \frac{C}{2}$$

$$f_1(x) = \frac{\varphi(x)}{2} - \frac{1}{2a} \int_{x_0}^x \psi(s) ds - \frac{C}{2}$$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds$$

Формула Коши-Гурса. Доказательство единственности решения задачи Коши.

Теорема существования и единственности решения задачи Коши.

Пусть $\varphi(x) \in C^2(E)$

$\psi(x) \in C^1(E)$ - Тогда функции $u(x,t)$, определяемые формулой Даламбера, т.е. $u(x,t) \in C^2(E \times \mathbb{R}_+)$ и $u(x,t)$, удовл.

(1) - (3) и $u(x,t)$ единственна.

Теорема устойчивости

Пусть $\varphi_i(x) \in C^2(E)$, $\psi_i(x) \in C^1(E)$, $i=1,2$.

и $\sup_{x \in E} |\varphi_1(x) - \varphi_2(x)| \leq \varepsilon$, $\sup_{x \in E} |\psi_1(x) - \psi_2(x)| \leq \varepsilon$. Тогда,

если $u_i(x,t)$ - реш. задач Коши (1)-(3) для $\varphi_i(x)$ и $\psi_i(x)$, то

$$\sup_{\substack{x \in E \\ t \in [0, T]}} |u_1(x,t) - u_2(x,t)| \leq \varepsilon(1+T)$$

Доказательство

$$|u_1(x,t) - u_2(x,t)| \leq \frac{|\varphi_1(x+at) - \varphi_2(x+at)|}{2} + \frac{|\varphi_1(x-at) - \varphi_2(x-at)|}{2} + \frac{1}{2a} \int_{x-at}^{x+at} |\psi_1(s) - \psi_2(s)| ds \leq \varepsilon + \varepsilon t$$

Характеристики уравнения в частных производных

$$(4) a_{11}(x,y) u_{xx} + 2a_{12}(x,y) u_{xy} + a_{22}(x,y) u_{yy} + F_1(x,y,u,u_x,u_y) = 0$$

$$a_{11}(x,y) dy^2 - 2a_{12}(x,y) dx dy + a_{22}(x,y) (dx)^2 = 0$$

Первые интегралы - кривые (4)

$$a^2 u_{xx} - u_{tt} = 0$$

$$a^2 dt^2 - dx^2 = 0$$

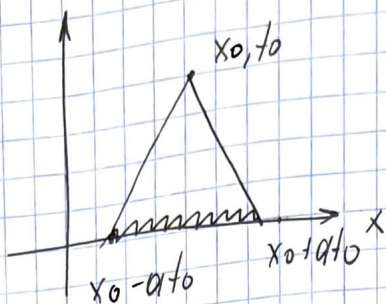
$$(adt - dx) \cdot (adt + dx) = 0$$

$$at - x = C \quad at + x = C$$

Характеристики

$$\begin{matrix} x+at \\ x-at \end{matrix}$$

$$u(x_0, t_0) = \frac{u(x_0 + at_0) + u(x_0 - at_0)}{2} + \frac{1}{2a} \int_{x_0 - at_0}^{x_0 + at_0} u(s) ds$$

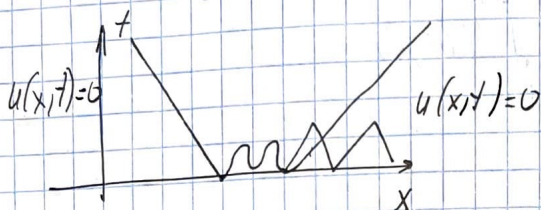


Задача Коши для уравнения теплопроводности

$$u_t = a^2 u_{xx}, \quad -\infty < x < \infty, t > 0$$

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

$$u(x, t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi a^2 t}} \exp\left(-\frac{(x-s)^2}{4a^2 t}\right) \varphi(s) ds$$



Задача N°13 (Удобенков)

$$u_{tt} = a^2 u_{xx}, \quad 0 < x, t < +\infty$$

$$u(0, t) = 0,$$

$$u(x, 0) = \varphi(x),$$

$$u_t(x, 0) = \psi(x)$$

$$u(x, t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha, \quad x-at \geq 0$$

$$u(x, t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{at-x}^{x+at} \psi(\alpha) d\alpha, \quad x-at < 0$$

II проверка задачи

$$u_{tt} = a^2 u_{xx}, \quad 0 < x, t < +\infty$$

$$u_x(0, t) = 0;$$

$$u(x, 0) = \varphi(x)$$

$$u_t(x, 0) = \psi(x)$$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(at-x)}{2} + \frac{1}{2a} \int_0^{x+at} \psi(\alpha) d\alpha + \frac{1}{2a} \int_0^{at-x} \psi(\alpha) d\alpha \quad x-at < 0$$

$$u(x,t) = \text{---} \text{---} \text{---} \text{---}, \quad x-at \geq 0$$

Lemma 1

$$\text{Given } v_{tt} = a^2 v_{xx}, \quad -\infty < x < \infty, t > 0,$$

$$v(x,0) = \varphi(x)$$

$$v_t(x,0) = \psi(x)$$

$$\varphi(-x) = -\varphi(x)$$

$$\psi(-x) = -\psi(x)$$

$$v(0,t) = 0$$

$$v(0,t) = \frac{\varphi(at) + \varphi(-at)}{2} + \frac{1}{2a} \int_{-at}^{at} \psi(\alpha) d\alpha = 0$$

Lemma 2

$$\text{Given } \varphi(-x) = \varphi(x),$$

$$\psi(-x) = \psi(x), \text{ mo}$$

$$v_x(x,t) = \frac{\varphi'(x+at) + \varphi'(x-at)}{2} + \frac{1}{2a} (\psi(x+at) - \psi(x-at))$$

$$v_x(0,t) = \frac{\varphi'(at) - \varphi'(-at)}{2} + \frac{1}{2a} (\psi(at) - \psi(-at)) = 0$$

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x, t < \infty \\ u_x(0,t) = 0 \\ u(x,0) = e^{-x^2} \\ u_t(x,0) = \cos x \end{cases} \quad \begin{cases} v_{tt} = a^2 v_{xx}, & -\infty < x \\ v(x,0) = e^{-x^2} \\ v_t(x,0) = \cos x \end{cases}$$

$$v(x,t) = \frac{e^{-(x+at)^2} + e^{-(x-at)^2}}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \cos \alpha d\alpha$$

$$\begin{cases} u_{2tt} = a^2 u_{2xx} \\ u_2(0,t) = g(t) \\ u_2(x,0) = 0 \\ u_{2t}(x,0) = 0 \end{cases} \quad 0 < x, t < \infty$$

$$u_2(x,t) = f(x-at) = g(t - \frac{x}{a})$$

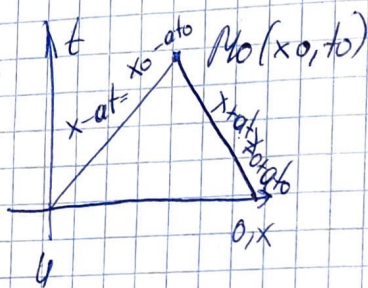
Если $t - \frac{x}{a} < 0$, то $g(t - \frac{x}{a}) = 0$

Если $t - \frac{x}{a} > 0$, $u_2(t) = g(t)$

$$u_2(x,t) = \begin{cases} 0, & x-at > 0 \\ g(t - \frac{x}{a}), & x-at < 0 \end{cases}$$

Для нахождения

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

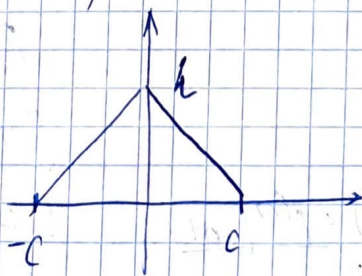


$$u(M_0) = u(x_0, t_0) = \varphi(p) + \varphi(q) + \frac{1}{2a} \int_{pq} \varphi(x) dx$$

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

$$\varphi(x) = h \left(1 - \frac{|x|}{c} \right)$$

$$\psi(x) = 0$$

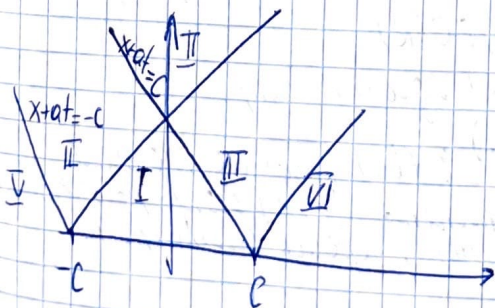


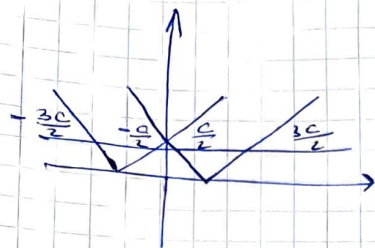
$$u_I = \frac{h}{2} \left(1 - \frac{|x+at|}{c} + 1 - \frac{|x-at|}{c} \right)$$

$$u_{II} = \frac{h}{2} \left(1 - \frac{|x+at|}{c} \right)$$

$$u_{III} = \frac{h}{2} \left(1 - \frac{|x-at|}{c} \right)$$

$$u_{IV}, u_V, u_{VI} = 0$$





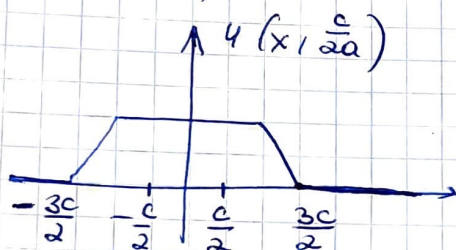
$$x < -\frac{3c}{2}, u=0$$

$$-\frac{3c}{2} < x < -\frac{c}{2}, u = \frac{h}{2} \left(1 - \frac{|x + \frac{c}{2}|}{\frac{c}{2}} \right)$$

$$-\frac{c}{2} < x < \frac{c}{2}, u = \frac{h}{2} \left(1 - \frac{|x + \frac{c}{2}|}{\frac{c}{2}} + 1 - \frac{|x - \frac{c}{2}|}{\frac{c}{2}} \right) = \frac{h}{2} \left(2 - \frac{x + \frac{c}{2}}{\frac{c}{2}} - \frac{(x - \frac{c}{2})}{\frac{c}{2}} \right) = \frac{h}{2}$$

$$\frac{c}{2} < x < \frac{3c}{2}, u = \frac{h}{2} \left(1 - \frac{|x - \frac{c}{2}|}{\frac{c}{2}} \right)$$

$$\frac{3c}{2} < x, u=0$$



Рассмотрим $\varphi^{(n)}(x) \Rightarrow \varphi(x), x \in \mathbb{R}$

$$\varphi(x) = \frac{1}{2a} \int_{-a}^a \varphi(x+d) dd$$

$$\varphi^{(n)} \Rightarrow \varphi$$

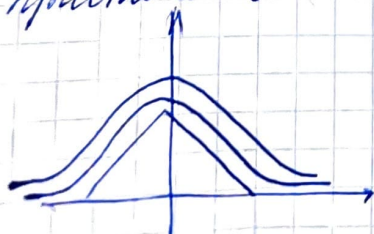
$$u^{(n)}(x,t) = \frac{\varphi^{(n)}(x+at) + \varphi^{(n)}(x-at)}{2} +$$

$$+ \varphi^{(n)}(x+at) - \varphi^{(n)}(x-at)$$

Если $n \rightarrow \infty$, то

$$\lim_{n \rightarrow \infty} u^{(n)}(x,t) = u(x,t)$$

* Если мы угодно можно приближить



Решение рассматривали

$$\begin{cases} u(x,0) = \varphi(x), \varphi(x) \in C^1(\mathbb{R}) \\ u_t(x,0) = \psi(x), \psi(x) \in C^1(\mathbb{R}) \end{cases}$$

Рассмотрим случай, когда $\varphi(x) \in C(\mathbb{R})$

$\varphi(x)$ непрерывно-испр.

$\varphi^{(n)} \Rightarrow \varphi(x)$, мы не можем придумать $\varphi(x)$ непрерывным, поэтому берем интеграл.

$$E(t) = \frac{1}{2} \int_0^l (\rho u_t^2 + \rho u_x^2) dx$$

Рассмотрим линейную задачу

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, 0 < t < T \\ u(0, t) = \mu_1(t) \\ u(l, t) = \mu_2(t) \\ u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u(0, t) = 0, \\ u(l, t) = 0 \\ u(x, 0) = \varphi(x), \\ u_t(x, 0) = \psi(x) \end{cases} \quad \begin{cases} x'' + \lambda x = 0 & P(0) = 1 \\ x(0) = 0 \\ x(l) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2; \quad x_n = \sin \frac{\pi n x}{l}$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n x}{l};$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin \frac{\pi n x}{l}$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin \frac{\pi n x}{l}$$

$$\varphi_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{\pi n x}{l} dx$$

$$\psi_n = \frac{2}{l} \int_0^l \psi(x) \sin \frac{\pi n x}{l} dx$$

$$\frac{d^2 u_n}{dt^2} = -\left(\frac{\pi n a}{l}\right)^2 u_n$$

$$u_n(0) = \varphi_n$$

$$u_n'(0) = \psi_n$$

$$u_n(t) = C_1 \cos \frac{\pi n a t}{l} + C_2 \sin \frac{\pi n a t}{l}$$

$$\begin{cases} u_n(0) = C_1 = \varphi_n \\ u_n'(0) = C_2 \frac{\pi n a}{l} = \psi_n \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos \frac{\pi n a t}{l} + \frac{l}{\pi n a} \psi_n \sin \frac{\pi n a t}{l} \right) \sin \frac{\pi n x}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\pi n x}{l}$$

Для непрерывности $u(x,t)$ нужно потребовать, чтобы

$$\sum_{n=1}^{\infty} \left(|\varphi_n| + \frac{l}{\pi n a} |\psi_n| \right) \text{ сходится}$$

$$\frac{\partial^2 u}{\partial t^2} \sim \sum_{n=1}^{\infty} \left(\frac{\pi n a}{l} \right)^2 \left(\cos \frac{\pi n a t}{l} + \varphi_n + \frac{l}{\pi n a} \psi_n \sin \frac{\pi n a t}{l} \right) \sin \frac{\pi n x}{l}$$

$$\sum_{n=1}^{\infty} \left(n^2 |\varphi_n| + \frac{l}{\pi n a} \cdot n |\psi_n| \right) \text{ сходится}$$

$$\sum_{n=1}^{\infty} n^k |\varphi_n|, k=0,1,2$$

$$\sum_{n=1}^{\infty} n^p |\varphi_n|, p=-1,0,1$$

"Сильные достаточные условия:"

$$\varphi(x) \in C^2[0, l]$$

$$\varphi(0) = \varphi(l)$$

$$\varphi'(0) = \varphi'(l)$$

$\varphi'''(x)$ — непрерывно

$$\varphi(x) \in C^1[0, l]$$

$\varphi''(x)$ непрерывно

Докажем теорему единственности

$$\begin{cases} V = u_1 - u_2 \\ V_t = a^2 V_{xx}, 0 < x < l, t > 0 \\ V(0,t) = 0, V(l,t) = 0 \\ V(x,0) = 0, V_t(x,0) = 0 \end{cases}$$

$$E(t) = \frac{1}{2} \int_0^l (\rho \bar{u}_t^2 + T \bar{u}_x^2) dx \quad u = v$$

$$a^2 = \frac{T}{\rho}$$

$$\begin{aligned} \frac{dE}{dt} &= \int_0^l (\rho \bar{u}_t \bar{u}_{tt} + T \bar{u}_x \bar{u}_{xt}) dx = \int_0^l \rho \bar{u}_t \bar{u}_{tt} dx + \int_0^l T \bar{u}_x \bar{u}_{xt} dx = \\ &= \int_0^l \rho \bar{u}_t \bar{u}_{tt} dx + T \bar{u}_x \bar{u}_t \Big|_0^l - \int_0^l T \bar{u}_{tx} \bar{u}_x dx = \end{aligned}$$

$$\int_0^l T \bar{u}_x \bar{u}_{xt} dx = \int_0^l \bar{u}_t (\rho \bar{u}_{tt} - T \bar{u}_{xx}) dx = 0$$

$$\frac{dE}{dt} \Big|_t = 0 \quad E(t) = E(0) = 0$$

$$\frac{1}{2} \int_0^l (\rho \bar{u}_t^2 + T \bar{u}_x^2) dx = 0$$

$$\begin{cases} \bar{u}_t = 0 \\ \bar{u}_x = 0 \Rightarrow v = 0 \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), \\ u(0,t) = 0; u(l,t) = 0 \\ u(x,0) = 0, u_t(x,0) = 0 \\ u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n x}{l}; \end{cases}$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{\pi n x}{l}$$

$$f_n = \frac{2}{l} \int_0^l f(x,t) \sin \frac{\pi n x}{l} dx$$

$$\begin{cases} \frac{d^2 u_n}{dt^2} + \left(\frac{\pi n a}{l} \right)^2 u_n = f_n(t); \\ u_n(0) = 0 \\ u_n'(0) = 0 \end{cases}$$

$$u_n(t) = \int_0^t v(t-\tau) f_n(\tau) d\tau \quad (1)$$

$$\begin{cases} \frac{d^2 v}{dt^2} + \left(\frac{\pi n a}{l}\right)^2 v = 0 \\ v(0) = 0 \quad v'(0) = 1 \end{cases}$$

← решаем это; потому подготовившим решение $v(t)$, и получаем

$$v(t) = \frac{l}{\pi n a} \sin \frac{\pi n a t}{l}$$

$$u_n(t) = \int_0^l \frac{l}{\pi n a} \sin \frac{\pi n a (t-\tau)}{l} f_n(\tau) d\tau$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{l}{\pi n a} \int_0^l \sin \frac{\pi n a (t-\tau)}{l} f_n(\tau) d\tau \right) \sin \frac{\pi n x}{l}$$

- 1-38
- 2-10
- 3-25
- 4-20
- 7-8-15"
- 5-6-4"
- 3-4-3"

20 января экзамен

Задача №13. (Решено)

Постановка задачи: найти решение задачи для уравнения колебаний на полупрямой.

Рассмотрим первую краевую задачу на полупрямой.

Найти $u(x,t)$ т.ч.

$$\begin{cases} u_{tt} = a^2 u_{xx}, t \geq 0 \\ u(0,t) = 0 \\ u(x,0) = \varphi(x), x \geq 0 \\ u_t(x,0) = \psi(x), x \geq 0 \end{cases}$$

Рассмотрим ф-ии $\Phi(x) = \begin{cases} \varphi(x), x \geq 0 \\ -\varphi(-x), x \leq 0 \end{cases}$, $\Psi(x) = \begin{cases} \psi(x), x \geq 0 \\ -\psi(x), x \leq 0 \end{cases}$

$$\varphi(0) = 0, \psi(0) = 0$$

Например,



$$v(x,t) \begin{cases} v_{tt} = a^2 v_{xx} & -\infty < x < \infty \\ & 0 < t < \infty \end{cases}$$

$$\begin{cases} v(x,0) = \Phi(x) \\ v_t(x,0) = \Psi(x) \end{cases}$$

$$v(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(s) ds$$

$$v(0,t) = v(0,t) = \frac{\Phi(at) + \Phi(-at)}{2} + \frac{1}{2a} \int_{-at}^{at} \Psi(s) ds$$

$$u(x,t) = \begin{cases} x-at \geq 0, \\ \frac{\Psi(x+at) + \Psi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(s) ds \\ x-at < 0, \\ \frac{\Psi(x+at) - \Psi(at-x)}{2} + \frac{1}{2a} \int_{at-x}^{x+at} \Psi(s) ds \end{cases}$$

$$u(x,t) = \frac{\Psi(x+at) - \Psi(at-x)}{2} + \frac{1}{2a} \int_0^{x+at} \Psi(s) ds + \frac{1}{2a} \int_0^{at-x} (-\Psi(s)) ds =$$

$$\frac{x-at}{2a} \int_0^{at-x} \Psi(s) ds$$

Вспомогательная задача

$$\begin{cases} u_{tt} = a^2 u_{xx} & x > 0 \\ & t > 0 \\ u_x(0,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & x \geq 0 \\ u_t(x,0) = \psi(x), & x \geq 0 \end{cases}$$

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x), & x \leq 0 \end{cases}, \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(x), & x \leq 0 \end{cases}$$

$$\varphi'(0) = 0$$

$$\psi'(0) = 0$$

$$\begin{cases} u_{tt} = a^2 u_{xx}, & x > 0 \\ t > 0 \end{cases}$$

$$u(x, 0) = \phi(x), \quad -\infty < x < \infty$$

$$u_t(x, 0) = \psi(x), \quad -\infty < x < \infty$$

$$\phi \in C^2(\bar{E}^+) \quad \psi \in C^1(\bar{E}^+)$$

$$u(x, t) = v(x, t) \quad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$

$$v_x(x, t) = \frac{\phi'(x+at) + \phi'(x-at)}{2} + \frac{1}{2a} [\psi(x+at) - \psi(x-at)] =$$

$$= \frac{\phi'(at) + \phi'(-at)}{2} + \frac{1}{2a} [\psi(at) - \psi(-at)]$$

Производные четных функций являются нечетными функциями.
Производные нечетных функций — четные.

$$u(x, t) = \begin{cases} \frac{\phi(x+at) + \phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds, & x-at > 0 \\ \frac{\phi(x+at) + \phi(at-x)}{2} + \frac{1}{2a} \int_0^{x+at} \psi(s) ds + \frac{1}{2a} \int_{x-at}^0 \psi(-\tilde{s}) d\tilde{s} = \\ \left\{ -\tilde{s} = s \right\} = \frac{1}{2a} \int_0^{at-x} \psi(s) ds = \frac{\phi(x+at) + \phi(at-x)}{2} + \frac{1}{2a} \int_0^{x+at} \psi(s) ds + \\ + \frac{1}{2a} \int_0^{at-x} \psi(s) ds, & x-at \leq 0 \end{cases}$$

Задача Коши для неоднородного уравнения
Найти $u(x, t)$ такую, что

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & -\infty < x < \infty \\ t > 0 \end{cases}$$

$$u(x, 0) = 0, \quad -\infty < x < \infty$$

$$u_t(x, 0) = 0, \quad -\infty < x < \infty$$

Теорема. Пусть функции $f(x, t)$ таковы, что $f(x, t) \in C(E \times \bar{E}^+)$ и $\frac{\partial f}{\partial x}(x, t) \in C(E \times \bar{E}^+)$ тогда существует единственная функция $u(x, t)$

$u(x, t) \in C^2(E \times \bar{E}^+)$ и $u(x, t)$ удовлетворяет (1)-(3)

Доказательство. Вспомогательные

$$u(x,t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(s,\tau) ds d\tau$$

$$u_t(x,t) = \frac{1}{2a} \int_0^t [f(x+a(t-\tau),\tau) + f(x-a(t-\tau),\tau)] d\tau$$

$$u_{tt}(x,t) = \frac{1}{2} [f(x,t) + f(x,t)] + \frac{a}{2} \int_0^t \left[\frac{\partial f}{\partial x}(x+a(t-\tau),\tau) - \frac{\partial f}{\partial x}(x-a(t-\tau),\tau) \right] d\tau$$

$$u_x(x,t) = \frac{1}{2a} \int_0^t [f(x+a(t-\tau),\tau) - f(x-a(t-\tau),\tau)] d\tau$$

$$u_{xx}(x,t) = \frac{1}{2a} \int_0^t \left[\frac{\partial f}{\partial x}(x+a(t-\tau),\tau) - \frac{\partial f}{\partial x}(x-a(t-\tau),\tau) \right] d\tau$$

сходимости.

Пусть

$$u_i(x,t) \text{ m.t. } u_i \in C^2(E \times E^+)$$

$$u_i(x,t) \text{ удовлетв. (1)-(3)}$$

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

$$\begin{cases} v_{tt} = a^2 v_{xx} & -\infty < x < \infty \\ v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases} \quad -\infty < x < \infty$$

По формуле для переменных односторонней гравитации Коши.

$$u_{tt} = a^2 u_{xx} + f(x,t), \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \varphi(x), \quad -\infty < x < \infty$$

$$u_t(x,0) = \psi(x), \quad -\infty < x < \infty$$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds + \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(s,\tau) ds d\tau$$

Задача №14 (Демидов)

Ливенский

$$u_t(0, z) + u_x(0, z) = u_t(z, 0) + u_x(z, 0)$$

$$u(x, t) = f(x+t) + g(x-t)$$

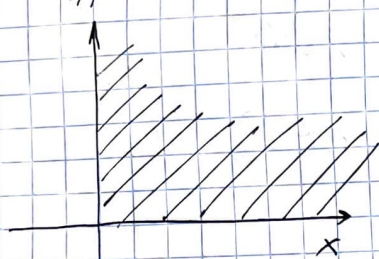
$$u_t = f'(x+t) - g'(x-t)$$

$$u_x = f'(x+t) + g'(x-t)$$

$$u_t(0, z) + u_x(0, z) = f'(z) - g'(-z) + f'(z) + g'(-z) = 2f'(z)$$

$$u_t(z, 0) + u_x(z, 0) = f'(z) + g'(z) + f'(z) - g'(z) = 2f'(z)$$

$$u_t = u_x, 0 \leq x, t \leq \infty$$



$$\begin{cases} u_{xy} = f(x, y), 0 \leq x, y \leq \infty \\ u|_{x=0} = \varphi_1(y) \\ u|_{y=0} = \varphi_2(x) \end{cases}$$

$$\int_0^x u_{yy}(y, y) dy = \int_0^x f(y, y) dy$$

$$u_y(x, y) - u_y(0, y) = \int_0^x f(y, y) dy$$

$$u_y(x, y) = u_y(0, y) + \int_0^x f(y, y) dy$$

$$\int_0^y u_{xy}(x, \eta) d\eta = \int_0^y f(x, \eta) d\eta$$

$$\begin{aligned} \int_0^y u_{xy}(x, \eta) d\eta &= \int_0^y u_{xy}(0, \eta) d\eta + \int_0^y \int_0^x f(y, \eta) dy d\eta \Rightarrow u(x, y) - u(x, 0) = \\ &= u(x, y) - u(0, 0) + \int_0^y \int_0^x f(y, \eta) dy d\eta \end{aligned}$$

$$Lu = u_{xy} + au_x + bu_y + cu = f(x, y)$$

$$a = a(x, y), b = b(x, y), c = c(x, y)$$

$$u|_{x=0} = \varphi_1(y)$$

$$u|_{y=0} = \varphi_2(x)$$

$$u_{xy} = -au_x - bu_y - cu + f(x, y)$$

$$u(x, y) = \varphi_2(x) + \varphi_1(y) - \varphi_1(0) - \int_0^y \int_0^x (au_y(y, \eta) + bu_x(y, \eta) + cu(y, \eta) - f(y, \eta)) dy d\eta$$

$$u_0(x, y) = 0$$

$$u_n(x, y) = u_1(x, y) - \int_0^y \int_0^x (a(u_{n-1})_x + b(u_{n-1})_y + cu_{n-1}) dy dx$$

$$\frac{\partial u_n}{\partial x} = \frac{\partial u_1}{\partial x} - \int_0^y (a(u_{n-1})_x + b(u_{n-1})_y + cu_{n-1}) dy$$

$$\frac{\partial u_n}{\partial y} = \frac{\partial u_1}{\partial y} - \int_0^x (a(u_{n-1})_x + b(u_{n-1})_y + cu_{n-1}) dx$$

$$\frac{\partial u_n}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = v$$

Легко показать, что это на самом деле так!

$$\frac{\partial u_n}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = w$$

$$u = - \int_0^y \int_0^x (av + bw + cu) dx dy + u_1(x, y)$$

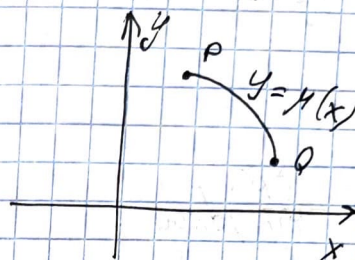
Задача Коши для уравнения Лапласа.

$$\Delta u = F(x, y)$$

$$y = \mu(x), \mu'(x) < 0$$

$$u|_{y=\mu(x)} = \varphi_0(x),$$

$$\frac{\partial u}{\partial y}|_{y=\mu(x)} = \varphi_1(x)$$



Рассмотрим оператор

$$Lu = u_{xy} + au_x + bu_y + cu$$

Сопреженный оператор

$$Mv = v_{xy} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv$$

$$Mv = v_{xy} - a \frac{\partial v}{\partial x} - b \frac{\partial v}{\partial y} - \frac{\partial a}{\partial x} v - \frac{\partial b}{\partial y} v + cv$$

$$M^*v = w_{xy} + \frac{\partial}{\partial x}(aw) + \frac{\partial}{\partial y}(bw) - \frac{\partial a}{\partial x} w - \frac{\partial b}{\partial y} w - \frac{\partial b}{\partial x} w + cw =$$

$$= w_{xy} + aw_x + bw_y + w \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) + cw$$

$$vLu - uMv = v \frac{\partial^2 u}{\partial x \partial y} + av \frac{\partial u}{\partial x} + bv \frac{\partial u}{\partial y} + cuv - \left(u \frac{\partial^2 v}{\partial x \partial y} - u \frac{\partial}{\partial x}(av) - u \frac{\partial}{\partial y}(bv) + cuv \right)$$

$$= v \frac{\partial^2 u}{\partial x \partial y} - u \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(avv) + \frac{\partial}{\partial y}(bvv)$$

сопряженный
от сопряженного
действия

$$v \frac{\partial^2 u}{\partial x \partial y} - u \frac{\partial^2 v}{\partial x \partial y} = \frac{1}{2} v \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2} u \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} v \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2} u \frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{1}{2} \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{1}{2} \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{1}{2} \frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial y} \right) + \frac{1}{2} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} \right) - \frac{1}{2} \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) + \frac{1}{2} \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right)$$

$$vLu - uMv = \frac{\partial}{\partial x} \left(\frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + auv \right) - \frac{\partial}{\partial y} \left(\frac{1}{2} u \frac{\partial v}{\partial x} - \frac{1}{2} v \frac{\partial u}{\partial x} - buv \right)$$

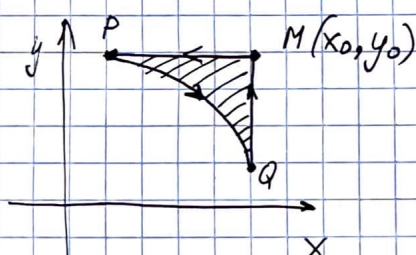
$$vLu - uMv = \frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y}$$

$$\tilde{Q} = \frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + auv$$

$$\tilde{P} = \frac{1}{2} u \frac{\partial v}{\partial x} - \frac{1}{2} v \frac{\partial u}{\partial x} - buv$$

Решения Грина

$$\oint_C \tilde{P} dx + \tilde{Q} dy = \iint_{\Omega} \left(\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y} \right) dx dy$$



$$\oint_C \tilde{P} dx + \tilde{Q} dy = \iint_{\Omega} (vLu - uMv) dx dy$$

$$C = \tilde{P} \tilde{Q} \tilde{M} \tilde{P}$$

$$\int_P^Q \tilde{P} dx + \int_Q^P \tilde{Q} dy (-1) + \int_Q^M \tilde{Q} dy + \int_P^M \tilde{P} dx$$

$$\int_P^Q \left(\frac{1}{2} u \frac{\partial v}{\partial x} - \frac{1}{2} v \frac{\partial u}{\partial x} - buv \right) dx +$$

$$\int_P^Q \tilde{Q} dy = \int_P^Q$$

$$\int_Q^M \left(\frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + auv \right) dy$$

$$\int_M^P = - \int_P^M = \int_P^M \left(\frac{1}{2} v \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial v}{\partial x} + buv \right) dx$$

$$\int y = \mu = \varphi_0(x)$$

$$\int \frac{\partial u}{\partial y} y = \mu = \varphi_1(x)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x} = \varphi_0'(x)$$

$$\frac{\partial u}{\partial x} = \varphi_0' - \varphi_1(\mu')$$

$$\int_Q^M \tilde{Q} dy = \int_Q^M \left(\frac{1}{2} v \frac{\partial v}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + a u v \right) dy = \int_Q^M \frac{1}{2} v du - \int_Q^M \frac{1}{2} u \frac{\partial v}{\partial y} dy -$$

$$- \int_Q^M \frac{1}{2} u \frac{\partial v}{\partial y} dy + \int_Q^M a u v dy$$

$$\int_Q^M \tilde{Q} dy = \frac{1}{2} u v \Big|_Q^M - \int_Q^M \frac{1}{2} u \frac{\partial v}{\partial y} dy + \int_Q^M a u v dy$$

$$\int_P^M \tilde{P} dx = \frac{1}{2} u v \Big|_P^M + \int_P^M u \left(-\frac{\partial v}{\partial x} + b v \right) dx$$

$$\iint_R (vLu - uMv) dx dy = \int_P^Q \tilde{P} dx + \tilde{Q} dy + \frac{1}{2} u v \Big|_Q^M + \frac{1}{2} u v \Big|_P^M +$$

$$+ \int_Q^M u \left(-\frac{\partial v}{\partial y} + a v \right) dy + \int_P^M u \left(-\frac{\partial v}{\partial x} + b v \right) dx$$

Нормализованные условия на v :

$$\begin{cases} Mv=0 \\ -\frac{\partial v}{\partial y} + av \Big|_{x=x_0}=0 \end{cases} \quad \swarrow \text{обобщенное} \\ \text{гипер-ое уравнение}$$

$$\begin{cases} -\frac{\partial v}{\partial x} + bv \Big|_{y=y_0}=0 \\ v(x_0, y) = e^{\int_{y_0}^y a(x_0, \eta) d\eta} + \int_P^M u \left(-\frac{\partial v}{\partial x} + b v \right) dx \end{cases}$$

$$v(x, y_0) = e^{\int_{x_0}^x b(y, y_0) dy}$$

$$v(x_0, y_0) = 1$$

$$\iint_R vLu dx dy = \int_P^Q \tilde{P} dx + \tilde{Q} dy + u(M) - \frac{1}{2} u(P) - \frac{1}{2} u(Q)$$

$$\iint_R v f(x, y) dx dy + \frac{1}{2} u(P) + \frac{1}{2} u(Q) = \Phi$$

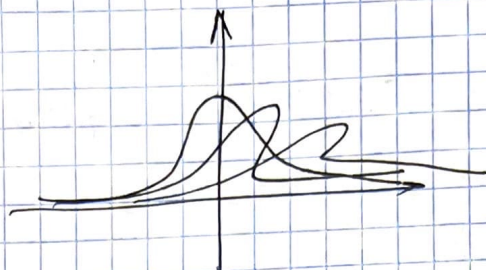
$$\Phi = \int_P^Q \tilde{P} dx + \tilde{Q} dy$$

Решение ищется

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

$$\frac{dt}{1} = \frac{dx}{u}, x - ut = C$$

$$u(x,t) = f(x-ct)$$



Уравнение Бюргера (1948г.)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}$$

$$u = \psi_x$$

$$\frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x^2} = D \frac{\partial^3 \psi}{\partial x^3}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial x} + \frac{1}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 - D \frac{\partial^2 \psi}{\partial x^2} \right) = 0$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 = D \frac{\partial^2 \psi}{\partial x^2}$$

Замена Коупа - Холгра.

$$\psi_t = -2D \frac{v_t}{v} \quad \psi = -2D \ln v$$

$$\psi_x = -2D \frac{v_x}{v} \quad v_{xx} = -2D \frac{v_{xx}v - v_x^2}{v^2}, \text{ подставим.}$$

$$-2D \frac{v_t}{v} + \frac{1}{2} \left(-2D \frac{v_x}{v} \right)^2 = -2D \frac{v_{xx}v - v_x^2}{v^2} D.$$

$$-2D \frac{v_t}{v} + 2D^2 \frac{v_x^2}{v^2} = -2D^2 \frac{v_{xx}}{v} + 2D^2 \frac{v_x^2}{v^2}$$

Задача N°14. (Демидов)

Средствование и единственность решения первой краевой задачи для уравнения колебаний.

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l, \quad 0 < t \leq T \quad (1)$$

$$u(0,t) = 0, \quad 0 \leq t \leq T \quad (2)$$

$$u(l,t) = 0, \quad 0 \leq t \leq T \quad (3)$$

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq l \quad (4)$$

$$u_t(x,0) = \psi(x), \quad 0 \leq x \leq l \quad (5)$$

$$\Pi = \{(x,t), \quad 0 \leq x \leq l, \quad 0 \leq t \leq T\}$$

Теорема существования

Пусть функции $\varphi(x), \psi(x)$ таковы, что:

$$\varphi(x) \in C^3[0, l], \psi(x) \in C^2[0, l], \varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0,$$

$\psi(0) = \psi(l) = 0$. Тогда существует функция $u(x, t)$ таковы, что $u(x, t) \in C^2(\Pi)$ и $u(x, t)$ удовлетворяет (1)-(5).

Разделение

$$\tilde{u}(x, t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t)$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, X(l) = 0 \end{cases} \quad T''(t) + a^2 \lambda T(t) = 0$$

Собственные значения

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, n=1, 2, \dots$$

$$X_n(x) = \tilde{C}_n \sin \frac{\pi n}{l} x$$

$$T_n(t) = \tilde{b}_n \cos \frac{\pi n}{l} at + \tilde{c}_n \sin \frac{\pi n}{l} at$$

$$\tilde{u}_n'(x, t) = X_n(x)T_n(t) =$$

$$= (\tilde{b}_n \cos \frac{\pi n}{l} at + \tilde{c}_n \sin \frac{\pi n}{l} at) \sin \frac{\pi n}{l} x$$

$$u(x, t) = \sum_{n=1}^{\infty} \tilde{u}_n(x, t)$$

$$\varphi_n = \frac{2}{l} \int_0^l \varphi(s) \sin \frac{\pi n}{l} s ds \quad n=1, 2, \dots$$

$$\psi_n = \frac{2}{l} \int_0^l \psi(s) \sin \frac{\pi n}{l} s ds$$

$$u(x, 0) = \sum_{n=1}^{\infty} \tilde{b}_n \cdot \sin \frac{\pi n}{l} x =$$

$$= \varphi(x) \Rightarrow \tilde{b}_n = \varphi_n$$

$$u(x, t) = \sum_{n=1}^{\infty} \left(\tilde{b}_n \left(-\frac{\pi n a}{l}\right) \sin \frac{\pi n a t}{l} + \tilde{c}_n \frac{\pi n a t}{l} \cos \frac{\pi n a t}{l} \right) \sin \frac{\pi n}{l} x$$

$$u(x, 0) = \sum_{n=1}^{\infty} \tilde{c}_n \frac{\pi n a t}{l} \sin \frac{\pi n}{l} x = \psi(x)$$

$$\tilde{c}_n = \frac{l}{\pi n a} \frac{2}{l} \int_0^l \psi(s) \sin \frac{\pi n}{l} s ds = \frac{l}{\pi n a} \psi_n$$

$$u(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos \frac{\pi n}{l} at + \frac{l}{\pi n a} \psi_n \sin \frac{\pi n}{l} at \right) \cdot \sin \frac{\pi n}{l} x$$

$$\varphi_n = \frac{2}{l} \int_0^l \varphi(s) \sin \frac{\pi n}{l} s ds = \frac{2}{l} \varphi(s) \left(-\frac{l}{\pi n} \cos \frac{\pi n}{l} s\right) \Big|_0^l + \frac{2}{\pi n} \int_0^l \varphi'(s) \cos \frac{\pi n}{l} s ds =$$

$$= \frac{2}{\pi n} \varphi'(s) \left(\frac{l}{\pi n}\right) \sin \frac{\pi n}{l} s \Big|_0^l - \frac{2l}{(\pi n)^2} \int_0^l \varphi''(s) \sin \frac{\pi n}{l} s ds = -\frac{2l}{(\pi n)^2} \int_0^l \varphi''(s) \sin \frac{\pi n}{l} s ds$$

$$\sin \frac{\pi n}{l} s ds = 0 - \frac{2l}{(\pi n)^2} \varphi''(s) \left(-\frac{l}{\pi n}\right) \cos \frac{\pi n}{l} s \Big|_0^l - \frac{2l^2}{(\pi n)^3} \int_0^l \varphi'''(s) \cos \frac{\pi n}{l} s ds$$

$$\varphi_n = \frac{-2l}{(\pi h)^2} \int_0^l \varphi''(s) \sin \frac{\pi h s}{l} ds$$

$$\varphi_n = \frac{-2l^2}{(\pi h)^3} \int_0^l \varphi'''(s) \cos \frac{\pi h s}{l} ds$$

$$\varphi_n(s) = \frac{-2l}{(\pi h)^2} \int_0^l \varphi''(s) \sin \frac{\pi h}{l} s ds$$

$f(x) \in C[0, l]$ $\xleftarrow{\text{Теорема Ейлера}}$ φ_n в квадрате, то пред с максимум значение энергии

$$f_n = \int_0^l f(s) \sin \frac{\pi h}{l} s ds$$

$$\hat{f}_n = \int_0^l f(s) \cos \frac{\pi h}{l} s ds$$

$$|\varphi_n| \leq \frac{C_1}{n^2} \quad |\varphi_n| \leq \frac{C_2}{n^2}$$

$$|\hat{u}_n(x, t)| \leq \frac{C_1}{n^3} + \frac{l}{\pi h a} C_2 \frac{1}{n^2} \quad \forall (x, t) \in \Pi$$

$\stackrel{=}{=} \frac{C_3}{n^2}$

$$u_x(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos \frac{\pi h}{l} a t + \frac{l}{\pi h a} \varphi_n \sin \frac{\pi h}{l} a t \right) \frac{\pi h}{l} \cos \frac{\pi h}{l} x = \frac{\partial \hat{u}_n}{\partial x}$$

$$\left| \frac{\partial u_n}{\partial x}(x, t) \right| \leq \left(\frac{C_1}{n^3} + \frac{l}{\pi h a} \frac{C_2}{n^2} \right) \frac{\pi h}{l} \leq \frac{C_4}{n^2}$$

$\forall (x, t) \in \Pi$. По теореме Вейерштрасса ряд из произведений энергии равномерно, значит можно почленно дифференцировать

$$u_{xx}(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos \frac{\pi h}{l} a t + \frac{l}{\pi h a} \varphi_n \sin \frac{\pi h}{l} a t \right) \left(-\left(\frac{\pi h}{l} \right)^2 \sin \frac{\pi h}{l} x \right)$$

$$\left| \frac{\partial^2 \hat{u}_n(x, t)}{\partial x^2} \right| \leq \left(|\varphi_n| + \frac{l}{\pi h a} |\varphi_n| \right) \left(\frac{\pi h}{l} \right)^2 \leq C_5 (|\varphi_n| n^2 + |\varphi_n| n) \leq$$

$$\varphi_n = \underbrace{\frac{-2l^2}{(\pi h)^3} \int_0^l \varphi'''(s) \cos \frac{\pi h s}{l} ds}_{\varphi_n'''} \quad , \quad \varphi_n = \underbrace{\frac{-2l}{(\pi h)^2} \int_0^l \varphi''(s) \sin \frac{\pi h s}{l} ds}_{\varphi_n''}$$

$$\sum_{n=1}^{\infty} (\varphi_n''')^2 < \infty \quad \sum_{n=1}^{\infty} (\varphi_n'')^2 < \infty$$

$$\leq C_6 \left(\frac{1}{n^3} |\varphi_n'''| n^2 + \frac{1}{n^2} |\varphi_n''| n \right) \leq C_6 \left(\frac{1}{2n^2} + \frac{1}{2} |\varphi_n'''|^2 + \frac{1}{2n^2} + \frac{|\varphi_n''|^2}{2} \right)$$

По теореме Вейерштрасса ряд, составленный из функций равномерно, значит почленно дифференцирование

справедливо, существуют вольные и свободные по x непрерывные,
 при x непрерывно и непрерывно функции непрерывны.

$$u(x,t) \in C^2(\Pi)$$

$$u(x,0) = \sum_{h=1}^{\infty} \varphi_h \sin \frac{\pi h}{\ell} x = \varphi(x)$$

$$u_t(x,0) = \sum_{h=1}^{\infty} \frac{1}{\pi h} \varphi_h \frac{\pi h}{\ell} \sin \frac{\pi h}{\ell} x = \psi(x)$$

Теорема существования

Пусть заданы $u_i(x,t)$ такие, что $i=1,2$

$u_i(x,t) \in C^2(\Pi)$ и выполняются граничные условия

$$\frac{\partial^2 u_i}{\partial t^2} = a^2 \frac{\partial^2 u_i}{\partial x^2} + f(x,t) \quad \begin{matrix} 0 < x < \ell \\ 0 < t \leq T \end{matrix}$$

$$u_i(0,t) = \mu_1(t), \quad 0 \leq t \leq T,$$

$$u_i(\ell,t) = \mu_2(t), \quad 0 \leq t \leq T,$$

$$u_i(x,0) = \varphi(x), \quad 0 \leq x \leq \ell,$$

$$\frac{\partial u_i}{\partial t}(x,0) = \psi(x), \quad 0 \leq x \leq \ell, \text{ тогда}$$

$$u_1(x,t) = u_2(x,t) \text{ где } (x,t) \in \Pi.$$

Разность

Разностью будем называть $v(x,t) = u_1(x,t) - u_2(x,t)$.

$v(x,t) \in C^2(\Pi)$ и все условия

$$v_{tt} = a^2 v_{xx}$$

$$v(0,t) = 0, \quad 0 \leq t \leq T,$$

$$v(\ell,t) = 0, \quad 0 \leq t \leq T,$$

$$v(x,0) = 0, \quad 0 \leq x \leq \ell,$$

$$v_t(x,0) = 0, \quad 0 \leq x \leq \ell$$

$$E(t) = \frac{1}{2} \int_0^{\ell} [a^2 (v_x(x,t))^2 + (v_t(x,t))^2] dx, \quad 0 \leq t \leq T$$

$$\frac{dE}{dt} = \int_0^{\ell} [a^2 v_x(x,t) v_{xt}(x,t) + v_t(x,t) v_{tt}(x,t)] dx$$

$$= a^2 v_x(x,t) v_t(x,t) \Big|_0^l + \int_0^l a^2 [v_{xx}(x,t) v_t(x,t) + v_t(x,t) v_{tt}(x,t)] dx =$$

$$= a^2 v_x(l,t) v_t(l,t) - a^2 v_x(0,t) v_t(0,t) + \int_0^l v_t(x,t) [-a^2 v_{xx} + v_{tt}(x,t)] dx =$$

$$= 0$$

$$\frac{dE}{dt}(t) = 0$$

$$E(t) = \text{const}, t \in [0, T]$$

$$E(0) = \frac{1}{2} \int_0^l [a^2 (v_x(x,0))^2 + (v_t(x,0))^2] dx = 0 \quad (\text{из I и II краевых условий}), \text{ возвращаясь к первоначальному заданию, имеем}$$

$$v_x(x,t) = v_t(x,t) = 0 \quad \forall t$$

$v(x,t) = \text{const}$, а поскольку имеем начальные краевые условия, получаем,

что это 0.

Для любой задачи гарантируется успех.

Лекция №15 (Ильиников)

$$\begin{cases} u_t + uu_x = 0 \frac{\partial^2 u}{\partial x^2}, & -\infty < x < +\infty, t > 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$

$$u = -20 \ln v, \quad u = -20 (\ln v)_x$$

$$\begin{cases} \frac{\partial v}{\partial t} = 0 \frac{\partial^2 v}{\partial x^2}, & -\infty < x < +\infty \\ v|_{t=0} = \Phi(x) \end{cases}$$

$$\Phi(x) = e^{-x^2}$$

$$e^{-x^2} = -20 (\ln \Phi)_x$$

$$-\frac{1}{20} \int_0^x e^{-\eta^2} d\eta = \ln \Phi(x)$$

$$v|_{t=0} = e^{-\frac{\sqrt{\pi}}{40} \cdot \text{erf}(x)}$$

~~$$\begin{aligned} \varphi(x) &= -20 (\ln \Phi)_x \\ (\ln \Phi)_x &= -\frac{\varphi(x)}{20} \\ \ln \Phi &= -\frac{1}{20} \int \varphi(\eta) d\eta \\ \varphi(x) &= e^{-\frac{1}{20} \int_{x_0}^x \varphi(\eta) d\eta} \end{aligned}$$~~

$$\varphi(x) = -20 (\ln \Phi)_x$$

$$(\ln \Phi)_x = -\frac{\varphi(x)}{20}$$

$$\ln \Phi = -\frac{1}{20} \int_0^x \varphi(\eta) d\eta$$

$$\Phi(x) = e^{-\frac{1}{20} \int_{x_0}^x \varphi(\eta) d\eta}$$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\pi t}} e^{-\frac{\sqrt{\pi}}{40} \sigma t f(y)} dy$$

Уравнение Корнелиуса-го Фруа КГБ

$$u_t + 6uu_x + u_{xxx} = 0$$

$$x' = kx, t' = \eta t, u' = du$$

$$u = f(x-ct) = f(\eta)$$

$$u_t = -cf'_\eta$$

$$u_x = f'_\eta$$

$$u_{xxx} = f'''_{\eta\eta\eta}$$

$$-cf'_\eta + f'_\eta \cdot 6 + f'''_{\eta\eta\eta} = 0$$

$$-\frac{cf}{2} + \frac{6}{2} f^2 + f''_{\eta\eta} = A$$

$$-cf f' + 3f^2 f' + f'$$

$$-cf f' + 3f^2 f' + f' f'' = A f' \quad (\text{интегрируем по } \eta)$$

$$-\frac{c}{2} f^2 + f^3 + \frac{1}{2} (f')^2 = A f + B$$

$$\text{Положим } A=0, B=0$$

$$-cf^2 + 2f^3 + (f')^2 = 0$$

$$(f')^2 = cf^2 - 2f^3$$

$$f' = \pm f \sqrt{c-2f}$$

$$\frac{df}{d\eta} = \pm f \sqrt{c-2f}$$

$$d\eta = \pm \frac{df}{f \sqrt{c-2f}}$$

$$\int c d\eta = \pm \int \frac{df}{f \sqrt{c-2f}}$$

$$\sqrt{c}(\eta - \eta_0) = \pm \int \frac{df}{f \sqrt{c-2f}}$$

$$\frac{2}{c}f = \frac{1}{\cosh^2 \alpha}$$

$$1 - \frac{1}{\cosh^2 \alpha} = \frac{\cosh^2 \alpha - 1}{\cosh^2 \alpha} = \frac{\sinh^2 \alpha}{\cosh^2 \alpha} = \tanh^2 \alpha$$

$$f = \frac{c}{2} \cdot \frac{1}{\cosh^2 \alpha}$$

$$df = \frac{c}{2} \cdot \frac{-2 \cdot \sinh \alpha}{\cosh^3 \alpha} d\alpha = -c \tanh \alpha \cdot \frac{1}{\cosh^2 \alpha} d\alpha$$

$$\sqrt{c}(y-x_0) = \pm \int \frac{c \tanh \alpha d\alpha}{\cosh^2 \alpha \cdot \frac{c}{2 \cosh^2 \alpha}}$$

$$\frac{\sqrt{c}}{2}(y-x_0) = \pm \int d\alpha$$

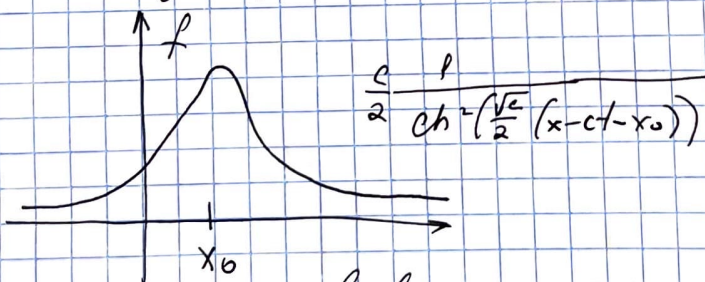
$$\cosh\left(\frac{\sqrt{c}}{2}(y-x_0)\right) = \cosh \alpha$$

$$\cosh^2\left(\frac{\sqrt{c}}{2}(y-x_0)\right) = \cosh^2 \alpha = \frac{c}{2f}$$

$$\frac{2}{c}f = \frac{1}{\cosh^2\left(\frac{\sqrt{c}}{2}(y-x_0)\right)}$$

$$u(x,t) = f(x-ct) = \frac{c}{2} \frac{1}{\cosh^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right)}$$

Нарисуем график этой функции.



Столкнуты две волны

Уравнение Коши-Власов - Мемма

Нелинейное уравнение теплопроводности.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right), & 0 < x < 1, t > 0 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

Уравнение Д'Аламбера

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) + f'(u)$$

$$k(u) = u^2$$

$$f(u) = u^3$$

$$u = x(x) \varphi(t)$$

$$x \varphi' = (xx')' \varphi^2$$

$$\frac{\varphi'}{\varphi^2} = \frac{(xx')'}{x} = -\lambda$$

$$\begin{cases} (xx')' + \lambda x \end{cases}$$

$$\begin{cases} x(0) = 0; x(1) = 0 \end{cases}$$

$$x = \alpha \tilde{x}$$

$$\alpha^2 (\tilde{x} \tilde{x}')' + \lambda \alpha \tilde{x} = 0$$

$$\alpha = \lambda$$

$$\begin{cases} (\tilde{x} \tilde{x}')' + \tilde{x} = 0 \end{cases}$$

$$\begin{cases} \tilde{x}(0) = 0; \tilde{x}(1) = 0 \end{cases}$$

$$\tilde{x} = y$$

$$\begin{cases} (yy')' + y = 0 \end{cases}$$

$$\begin{cases} y(0) = y(1) = 0 \end{cases}$$

$$\left(\frac{\sqrt{g} g'}{2\sqrt{g}} \right)' + \sqrt{g} = 0$$

$$y = F(x)$$

$$y = g$$

$$g'' + 2\sqrt{g} = 0$$

$$g'' = -2\sqrt{g}$$

$$\frac{g'}{g^2} = -1; \frac{d\varphi}{\varphi^2} = -dt; \varphi(t) = \frac{1}{t+t_0}$$

$$u(x,t) = \frac{F(x)}{t+t_0}$$

$$u(x,t) = \varphi_1 \sin \pi x e^{-\frac{(\pi a)^2 t}{t+t_0}} + \varphi_2 \sin 2\pi x e^{-\frac{4(\pi a)^2 t}{t+t_0}}$$

$$u(x,t) \rightarrow 0 \quad u(x,t) \sim \varphi_1 \sin \pi x e^{-\frac{(\pi a)^2 t}{t+t_0}}$$